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Effect of Surface Tension in the Measurement of the Average Normal Stress at the Exit of a Capillary Tube through an Analysis of the Capillary Jet

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Synopsis

The analysis of a stationary, rotationally symmetric liquid jet which leads to an expression for the average normal stress at the exit of a capillary tube is reexamined with particular attention to the effect of surface tension. The limiting case of a nearly cylindrical jet is compared with the analysis presented by Gavis and Middleman.

1. Introduction

There has been considerable interest in attempting to use measurements on the stationary liquid jet formed at the exit of a horizontal capillary tube to determine with some approximation the relation between two of the normal components of stress within the flow in the tube.[1–4] Several papers[1–4] neglect the effect of surface tension in treating this problem. More recently Gavis and Middleman[5–8] suggested that surface tension may not be negligible in many cases. Below we derive a more complete expression for the effect of surface tension which is compared with that of Gavis and Middleman in the limit of a nearly cylindrical jet.

2. Analysis

In what follows we make a momentum balance on a portion of the capillary jet.

Let us choose as our system of volume V the fluid contained in the jet between the exit plane of the tube, S_1 (assumed to be normal to the flow), and some plane S_2 parallel to S_1 and at a sufficient distance downstream that the velocity in the jet is no longer a function of axial position. This last condition makes it clear that the axis of the jet must be horizontal.[4] The bounding surface S of V can be divided into three parts: S_1, S_2, and S_f, which denotes the free surface of the jet between S_1 and S_2 (i.e., $S_f = S - S_1 - S_2$).

For a steady-state flow the stress equation of motion may be written*

$$0 = (t^{ij} - \rho v^i v^j)_{,j} + \rho f^i \tag{1}$$

where t^{ij} is the stress tensor, ρ is the density, v^i is the velocity vector, and f^i is the external body force vector per unit mass. Integrating eq. (1) over V and making use of Green's theorem[10] we obtain

$$0 = \int_S g_i^{\ M}[t^{ij} - \rho v^i v^j] n_j \, dS + \int_V g_i^{\ M} \rho f^i \, dV \tag{2}$$

Since the integral of a vector is not in general a vector, we make use of the shifter g_i^M defined[11] as

$$g_i^{\ M} = \delta_k^{\ K} \frac{\partial X^M}{\partial Z^K} \frac{\partial z^k}{\partial x^i} \tag{3}$$

where $\delta_k^{\ K} = 1$ if $K = k$ and $\delta_k^{\ K} = 0$ if $K \neq k$. The quantities X^M and Z^K represent the curvilinear and rectangular, Cartesian coordinates, respectively, of some point to which all of the vectors in the integrand are shifted. The quantities x^i and z^k represent, respectively, the curvilinear and rectangular Cartesian coordinates of each point in the field of integration. Here (2) n_1 is an outwardly directed unit vector normal to the closed surface S.

Let us consider our problem in cylindrical coordinates

$$\begin{aligned} x^1 &= \theta \\ x^2 &= z \\ x^3 &= r \end{aligned} \tag{4}$$

where z increases in the direction of flow. From eq. (3)

$$\begin{aligned} g_1^{\ 2} &= g_3^{\ 2} = 0 \\ g_2^{\ 2} &= 1 \end{aligned} \tag{5}$$

and the second component of eq. (2) becomes (if we take the external body force to represent the force of gravity)

$$0 = \int_S [t^{2j} - \rho v^2 v^j] n_j \, dS \tag{6}$$

Since the free surface of the jet is stationary

$$0 = \int_{S_f} t^{2j} n_j \, dS + \int_{S_1} \rho (v^2)^2 \, dS - \int_{S_2} \rho (v^2)^2 \, dS - \int_{S_1} t^{22} \, dS + \int_{S_2} t^{22} \, dS \tag{7}$$

The fluid is taken to be incompressible and the velocity profile at S_2 is assumed to be relatively uniform so that we may take the average of the square of v^2 to be the square of the average of v^2

$$\int_{S_2} \rho (v^2)^2 \, dS = \rho Q^2 / S_2 \tag{8}$$

Here Q indicates the volume flow rate.

* Latin indices indicate tensors with respect to coordinate transformations in 3-space; greek indices denote tensors with respect to surface coordinate transformations. Comma notation stands for covariant differentiation[9a] and the summation convention is employed throughout.

Let us examine the surface equations of motion.[12,13] We neglect inertial effects in the surface, the effects of the surface viscosities, the effect of gradients in surface tension, and the momentum exchange with adjacent phases due to mass transfer to obtain[13]

$$2Hn^i\sigma = -t^{ij}_{(1)}\, n_{(1)j} - t^{ij}_{(2)}\, n_{(2)j} \tag{9}$$

Here $n^j_{(K)}$ is a unit vector normal to the surface and outwardly directed into phase K, $t^{ij}_{(K)}$ is the stress tensor adjacent to the surface in phase K, σ is surface tension, n^i is a unit vector normal to the surface such that ($x_{;1}{}^r$, $x_{;2}{}^r$, n^r) have the same orientation as the tangents to the spatial coordinate curves[9a]

$$n_r = \epsilon^{\alpha\beta}\epsilon_{rst}x_{;\alpha}{}^s\, x_{;\beta}{}^t/2 \tag{10}$$

$$x_{;\alpha}{}^i = \partial x^i/\partial u^\alpha \tag{11}$$

The quantity ϵ_{rst} is defined as $g^{1/2}e_{rst}$, where g is the determinant of the matrix of g_{mn} and e_{rst} is the skew-symmetric relative tensor such that[9d] $e_{123} = 1$ (g_{mn} being the metric tensor in 3-space[9e]); $\epsilon^{\alpha\beta}$ is defined as $e^{\alpha\beta}/a^{1/2}$, where $e^{\alpha\beta}$ is the skew-symmetric relative surface tensor such that[9f] $e^{12} = 1$ (a being the determinant of the matrix of $a_{\alpha\beta}$). The mean curvature of the surface is[9b]

$$H = a^{\alpha\beta}b_{\alpha\beta}/2, \tag{12}$$

$a_{\alpha\beta}$ is the surface metric tensor,[9c]

$$a_{\alpha\beta} = g_{ij}x_{;\alpha}{}^i x_{;\beta}{}^j \tag{13}$$

and $b_{\alpha\beta}$ is a symmetric tensor in Gauss's formulae[9b]

$$b_{\alpha\beta} = -g_{mn}n_{,\alpha}{}^m x_{;\beta}{}^n \tag{14}$$

We observe that for the case we wish to consider the surface has no normal component of velocity and that to good approximation (i.e., neglecting any distortion by the external body force) the surface is one of revolution. Accordingly, the surface coordinates are chosen to be

$$u^1 = x^1 \tag{15}$$

and on the surface

$$x^3 = f(u^2) \tag{16}$$

From eq. (13) the components of the surface metric tensor are

$$\begin{aligned} a_{11} &= (f)^2 \\ a_{12} &= 0 \\ a_{22} &= 1 + (f')^2 \end{aligned} \tag{17}$$

Let us denote the liquid in the jet as phase 1 and the air surrounding the liquid jet as phase 2. Then in the coordinate system described above

$$n^i = n^i_{(2)} = -n^i_{(1)} \tag{18}$$

and from eq. (10)

$$n_1 = 0$$
$$n_2 = -f'/[1 + (f')^2]^{1/2} \tag{19}$$
$$n_3 = 1/[1 + (f')^2]^{1/2}$$

From eq. (14),

$$b_{11} = -f/[1 + (f')^2]^{1/2}$$
$$b_{12} = 0 \tag{20}$$
$$b_{22} = f''/[1 + (f')^2]^{1/2}$$

and we find the mean curvature of the surface from eq. (12) to be

$$H = \frac{ff'' - (f')^2 - 1}{2f[1 + (f')^2]^{3/2}} \tag{21}$$

The second component of eq. (9) becomes, in view of eq. (18),

$$2Hn^2\sigma = t_{(1)}{}^{2j}n_j - t_{(2)}{}^{2j}n_j \tag{22}$$

If we neglect any viscous effects in the surrounding air stream where the ambient pressure is p_0

$$(2H\sigma - p_0)\, n^2 = t_{(1)}{}^{2j}n_j \tag{23}$$

$$\frac{\sigma f'[-ff'' + (f')^2 + 1]}{f[1 + (f')^2]^2} + \frac{p_0 f'}{[1 + (f')^2]^{1/2}} = t_{(1)}{}^{2j}n_j \tag{24}$$

which can be used to evaluate the first term on the right of eq. (7).

In choosing the axial location of S_2 we specify that $\partial v^2/\partial x^2 = 0$ at S_2. For a Newtonian fluid or for a generalized Newtonian fluid (a special case of the Stokesian fluid[14] for which $\gamma = 0$ in Serrin's eq. (59.3) and which includes all of the common empirical models for non-Newtonian behavior)

$$t_{ij} = -pg_{ij} - \eta d_{ij} \tag{25}$$

$$\eta = \eta\,(\Pi_d),\ \Pi_d = d^{ij}d_{ij} \tag{26}$$

$$d_{ij} = (v_{i,j} + v_{j,i})/2 \tag{27}$$

this condition implies that

$$S_2 : t_{22} = -p \tag{28}$$

While this would not necessarily be true in the general case of a fluid described by the Coleman-Noll theory for simple fluids[15–17] the assumption is commonly made that eq. (28) is approximately obeyed in all cases of interest;[1,3] this assumption is usually stated as requiring the stresses at S_2 to be "relaxed." From the third component of eq. (9) we have

$$2Hn^3\sigma = t_{(1)}{}^{3j}n_j - t_{(2)}{}^{3j}n_j \tag{29}$$

The assumption $\partial v^2/\partial x^2 = 0$ at S_2 implies that $f' = f'' = 0$ and

$$S_2:\ n_1 = n_2 = 0,\ n_3 = 1,\ H = -1/(2f) \tag{30}$$

We neglect the viscous effects in the surrounding air stream and find

$$S_2:\ -\sigma/f = t_{(1)}{}^{33} + p_0 \tag{31}$$

Consistent with our assumption of a uniform profile in obtaining eq. (8), we take

$$S_2:\ t_{(1)}{}^{33} = -p \tag{32}$$

to conclude that

$$S_2:\ p = \sigma/R_2 + p_0 = -t_{22} \tag{33}$$

where R_2 is the radius of the jets at S_2.

Returning to eq. (7), from eqs. (8), (24), and (33) we have for the average normal stress acting at S_1

$$\overline{t_{zz}} = \frac{1}{S_1}\int_{S_1} t^{22}\, dS = \frac{2\pi}{S_1}\int_0^L \left\{\frac{\sigma f'[-ff'' + (f')^2 + 1]}{f[1 + (f')^2]^{3/2}} + p_0 f'\right\} f\, dz$$
$$+ \frac{\rho}{S_1}\int_{S_1} (v^2)^2\, dS - \frac{\rho Q^2}{S_1 S_2} - (p_0 + \sigma/R_2)\frac{S_2}{S_1} \tag{34}$$

where L is the value of the cylindrical coordinate z at S_2 (z is zero at S_1). Here we have rewritten the first integral on the right of eq. (7) making use of the expression for the differential element of area[9g]

$$dS = \sqrt{a}\, du^1\, du^2 \tag{35}$$

where

$$a = \det \|a_{\alpha\beta}\|$$

3. Comparison with Previous Results

If the effect of surface tension is neglected entirely in eq. (34), the first integral on the right becomes

$$\frac{2\pi}{S_1}\int_0^L p_0 f' f\, dz = \frac{p_0}{S_1}[S_2 - S_1] \tag{36}$$

and eq. (34) reduces to

$$\overline{t_{zz}} + p_0 = \frac{\rho}{S_1}\int_{S_1} (v^2)^2\, dS - \frac{\rho Q^2}{S_1 S_2} \tag{37}$$

which is in agreement with the expression proposed by Metzner, Houghton, Sailor, and White.[1-4] (Their normal component of stress is computed relative to atmospheric pressure.) As White[4] points out, Sakiadis[3] incorrectly considers a vertical jet.

For a nearly cylindrical jet $(f')^2 << 1$ and $f'' \doteq 0$. With this assumption the first integral on the right of eq. (34) becomes

$$\frac{2\pi}{S_1}\int_0^L [\sigma f' + p_0 f'f]dz = \frac{2\pi}{S_1}(R_2 - R_1)\,[\sigma + p_0(R_2 + R_1)]/2 \quad (38)$$

For most cases of practical importance $\sigma << p_0(R_2 + R_1)/2$

$$\frac{2\pi}{S_1}\int_0^L [\sigma f' + p_0 f'f]dz = \frac{p_0}{S_1}(S_2 - S_1) \quad (39)$$

If we assume the Poiseuille velocity profile at the exit of the tube[18] (this is justified only in the case of a Newtonian fluid),

$$\frac{S_1}{Q^2}\int_{S_1} (v^2)^2 dS = {}^4/_3 \quad (40)$$

and eq. (34) reduces to [in dimensionless from obtained by dividing through by $\rho Q^2/(S_1)^2$]

$$\frac{(S_1)^2}{\rho Q^2}[\bar{t}_{zz} + p_0] = -\frac{2}{(We)^2}\frac{R_2}{R_1} + {}^4/_3 - \frac{S_1}{S_2} \quad (41)$$

where

$$(We)^{-2} = \sigma(S_1)^2/(2\rho Q^2 R_1)$$

This is not quite the result presented by Gavis and Middleman[5] [their eq. (2)] for a Newtonian fluid:

$$\frac{(S_1)^2}{\rho Q^2}[\bar{t}_{zz} + \bar{p}] = -\frac{2}{(We)^2}\frac{R_2}{R_1} + {}^4/_3 - \frac{S_1}{S_2} \quad (42)$$

where

$$\bar{p} = \frac{1}{S_1}\int_{S_1} p\,dS \neq p_0 \quad (43)$$

If radial changes in pressure are neglected (which is consistent with the assumption of a nearly cylindrical jet), and if it is assumed that $t_{rr} = -p$ at S_1 (which is also consistent with the assumption of a nearly cylindrical jet for the case of a Newtonian fluid), from eqs. (29) and (43) we have

$$p \doteq p_0 + \sigma/R_1 \quad (44)$$

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Résumé

L'analyse d'un jet stationnaire, symétrique pour une rotation qui permet de calculer l'expression de la tension normale moyenne à la sortie d'un tube capillaire, est reexaminée en attachant une attention particulière à l'effet de la tension superficielle. On a pu montrer que l'expression donnée par Gavis et Middleman n'est valide que dans le cas limite d'un jet cylindrique.

Zusammenfassung

Die Analyse eines stationären rotationssymmetrischen Flüssigkeitsstrahls, die zu einem Ausdruck für die mittlere Normalspannung an der Ausflussöffnung einer Kapillarröhre führt, wird mit besonderer Berücksichtigung des Einflusses der Oberflächenspannung durchgeführt. Es wird gezeigt, dass der Ausdruck von Gavis und Middleman für den Grenzfall eines nahezu zylindrischen Strahles gültig ist.

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